情報通信を行なう際の要請
(What is required in communication ?)

- **効率的に (efficiently)**
  $\longrightarrow$ **情報理論 (Information Theory)**

- **確実に (certainly)**
  $\longrightarrow$ **符号理論 (Coding Theory)**

- **安全に (safely)**
  $\longrightarrow$ **暗号理論 (Cryptography)**

情報理論 (情報源符号化・情報量の理論)

(Information Theory, Source coding)

- **伝えるべき情報をより効率良く伝えるには**
  (How to communicate more efficiently)
- **「効率の良さ」を計る**
  (quantify (formulate) “efficiency”)
  - $\star$ **伝えるべき「情報の量」を計る**
    (quantify “information”)
  - $\star$ **伝える為の「手間」を計る**
    (quantify “cost to transmit”)

$\longrightarrow$ **Shannon**

「情報の量は伝えるのに必要な手間と一致」

(“quantity of information = minimum cost”)

情報の符号化 (Coding)

Analog data (continuous data)

$\Downarrow$ **sampling**

Digital data (discrete, finite)

$\uparrow$ **"情報 (information)" to be treated here**

(情報源, source)

$\Downarrow \longleftarrow$ **"符号化 (coding)" to be treated here**

Digital data for transmission (伝送用データ)

$\cdots$ **特定 (一般には少数) の種類の文字の列**

(a sequence of specific alphabets)

情報源符号化の定式化
(Formulation of source coding)

source (情報源) alphabet S : a finite set

$S^+ := \bigsqcup_{n \geq 1} S^n$: S の元の 1 個以上の列全体

ε : 空語 (the empty word), $S^0 := \{\varepsilon\}$

$S^* := \bigsqcup_{n \geq 0} S^n$: S の元の 0 個以上の列全体

$= S^+ \sqcup \{\varepsilon\}$

$w \in S^n$ に対し、$|w| := n$
(the length of a sequence)

情報源符号化の定式化
(Formulation of source coding)

code alphabet T **: a finite set**

(typically $\mathsf{T} = \{0, 1\}$**)**

$\mathcal{C} : S \longrightarrow \mathsf{T}^{+}$ **: 符号 (code)**

$w \in \mathrm{Im}\mathcal{C}$**: 符号語 (code-word)**

$\longrightarrow$ **文字列を並べて** $\mathcal{C}^{*} : S^{*} \longrightarrow \mathsf{T}^{*}$ **に延長**

(extended by concatnation)

符号への要請 (Requirement for good codes)

- Uniquely decodable (一意復号可能)？

- Furthermore, instantaneously decodable (瞬時復号可能)？

- Furthermore, efficient (効率的)？

一意復号可能でない例
(Ex. not uniquely decodable)

$$S = \{a, b, c\} \qquad T = \{0, 1\}$$

$$\mathcal{C} : \begin{cases} a \longmapsto 0 \\ b \longmapsto 01 \\ c \longmapsto 001 \end{cases}$$

「001」が ab か c か判らない
(cannot distinguish between “ab” and “c”)

$\longrightarrow$ **一意復号可能でない!!**
(NOT uniquely decodable!!)

瞬時復号可能でない例
(Ex. not instantaneously decodable)

$$S = \{a, b, c\} \qquad \mathcal{C} : \begin{cases} a \longmapsto 0 \\ b \longmapsto 01 \\ c \longmapsto 11 \end{cases}$$
$$T = \{0, 1\}$$

- **Uniquely decodable (一意復号可能ではある)**
- **"011⋯" $\Longrightarrow$ "ac⋯" or "bc⋯" ?**
  ("0111"$\Longrightarrow$"bc", "01111"$\Longrightarrow$"acc")
  $\longrightarrow$ **NOT instantaneously decodable**
  (瞬時復号可能でない)

瞬時復号可能な例

(Ex. instantaneously decodable)

$$S = \{a, b, c\} \qquad \mathcal{C} : \begin{cases} a \longmapsto 0 \\ b \longmapsto 10 \\ c \longmapsto 11 \end{cases}$$

$$T = \{0, 1\}$$

$\longrightarrow$ **How can one distinguish instantaneously decodable codes by looking only at $\mathcal{C}(S) = \{0, 10, 11\} \subset T^+$?**

符号への要請 (Requirement for good codes)

- **一意符号 (uniquely decodable code):**

$$\mathcal{C}^* : S^* \longrightarrow \mathsf{T}^* : \textbf{injective (単射)}$$

- **瞬時符号 (instantaneously decodable code):**

$$\mathcal{C}^*(\boldsymbol{x}) = \mathcal{C}(s)\boldsymbol{w} \Longrightarrow \boldsymbol{x} = s\boldsymbol{y}$$

(If the received sequence starts with $\mathcal{C}(s)$, the source sequence starts with s.)

- **効率が良い (efficient)**
  $\cdots$ **the lengths $|\mathcal{C}(s)|$ of code-words are small**

瞬時符号の性質
(Properties of instantaneously decodable codes)

- $\mathcal{C}$: instant. decodable $\Longrightarrow$ $\mathcal{C}$: uniq. decodable
- $\mathcal{C}$: instant. decodable
  $\Longleftrightarrow$ $\mathcal{C}$: prefix code (語頭符号)
  ($\mathcal{C}(s') = \mathcal{C}(s)x \Longrightarrow s' = s, x = \varepsilon$)

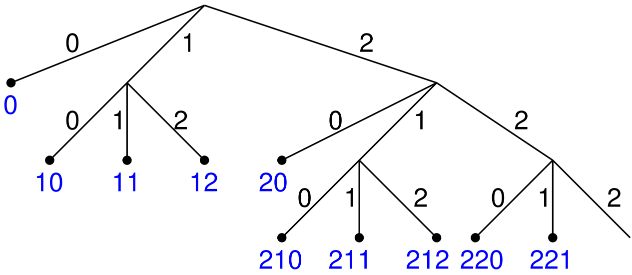
瞬時符号の作り方
(How to construct instant. decodable codes)

符号木 (code tree) を考えよう

符号木 (code tree)

Ex. $T = \{0, 1, 2\}$

$$\mathcal{C}(S) = \{0, 10, 11, 12, 20, 210, 211, 212, 220, 221\}$$

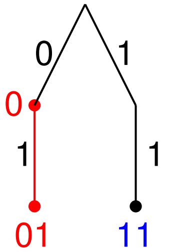


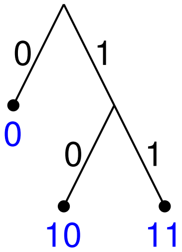
符号木と瞬時復号可能性

(Code trees and instantaneous decodablity)

Ex. $T = \{0, 1\}$

$\mathcal{C}(S) = \{0, 01, 11\}$ $\mathcal{C}(S) = \{0, 10, 11\}$





not instant. decodable **instant. decodable**

瞬時符号の効率

(Efficency of instantaneously decodable codes)

瞬時符号という条件を満たしつつ、
出来るだけ効率良くしたい

(Under the condition to be instant. decodable, make it as efficient as possible.)

The list of the lengths of the code-words

$$(|\mathcal{C}(s)|)_{s\in S} = (\mathcal{C}(s_1), \mathcal{C}(s_2), \ldots, \mathcal{C}(s_k))$$

is to be as "small" as possible.

Kraft の不等式 (Kraft's inequality)

$S = \{s_1, \ldots, s_k\}$, $\qquad \#T = r$ **(r-ary code)**

For a sequence $(\ell_1, \ldots, \ell_k)$ **of natural numbers,**

$\exists$ **an** r**-ary** **instant. decodable code** $\mathcal{C}$
with $|\mathcal{C}(s_i)| = \ell_i$ **($\forall i$)**

$$\Longleftrightarrow \sum_{i=1}^{k} \frac{1}{r^{\ell_i}} \leq 1$$

instant. decodable $\Longrightarrow$ uniq. decodable
(the converse is not true)
(Uniq. decodability is a weaker condition.)

If we allow uniq. decodable codes,
can we make the list of lengths more small ?

$\longrightarrow$ No!!

(For any uniq. decodable code,
there exists a instant. decodable code
with the same list of lengths.)

McMillan の不等式 (McMillan's inequality)

$S = \{s_1, \ldots, s_k\}, \qquad \#T = r$ **(r-ary code)**

For a sequence $(\ell_1, \ldots, \ell_k)$ **of natural numbers,**

$\exists$ **an** r**-ary** **uniq. decodable code** $\mathcal{C}$
with $|\mathcal{C}(s_i)| = \ell_i$ **(**$\forall i$**)**

$$\iff \sum_{i=1}^{k} \frac{1}{r^{\ell_i}} \leq 1$$