情報源符号化の定式化
(Formulation of source coding)

source (情報源) alphabet S : a finite set

$S^+ := \bigsqcup_{n \geq 1} S^n$: **S の元の 1 個以上の列全体**

ε : **空語 (the empty word),** $\quad S^0 := \{\varepsilon\}$

$S^* := \bigsqcup_{n \geq 0} S^n$: **S の元の 0 個以上の列全体**

$= S^+ \sqcup \{\varepsilon\}$

$w \in S^n$ **に対し、** $|w| := n$

(the length of a sequence)

情報源符号化の定式化
(Formulation of source coding)

code alphabet T **: a finite set**

(typically $\mathsf{T} = \{0, 1\}$**)**

$\mathcal{C} : S \longrightarrow \mathsf{T}^+$ **: 符号 (code)**

$w \in \mathrm{Im}\mathcal{C}$**: 符号語 (code-word)**

$\longrightarrow$ **文字列を並べて** $\mathcal{C}^* : S^* \longrightarrow \mathsf{T}^*$ **に延長**

(extended by concatnation)

符号への要請 (Requirement for good codes)

- **Uniquely decodable (一意復号可能) ?**
- **Furthermore, instantaneously decodable (瞬時復号可能) ?**
- **Furthermore, efficient (効率的) ?**

符号への要請 (Requirement for good codes)

- **一意符号 (uniquely decodable code):**

$$\mathcal{C}^* : S^* \longrightarrow \mathsf{T}^* : \textbf{injective (単射)}$$

- **瞬時符号 (instantaneously decodable code):**

$$\mathcal{C}^*(\boldsymbol{x}) = \mathcal{C}(s)\boldsymbol{w} \Longrightarrow \boldsymbol{x} = s\boldsymbol{y}$$

(If the received sequence starts with $\mathcal{C}(s)$,
the source sequence starts with s.)

- **効率が良い (efficient)**
  $\cdots$ **the lengths $|\mathcal{C}(s)|$ of code-words are small**

一意復号可能でない例
(Ex. not uniquely decodable)

$$S = \{a, b, c\}$$
$$T = \{0, 1\}$$

$$\mathcal{C} : \begin{cases} a \longmapsto 0 \\ b \longmapsto 01 \\ c \longmapsto 001 \end{cases}$$

「001」が ab か c か判らない
(cannot distinguish between "ab" and "c")

$\longrightarrow$ 一意復号可能でない!!
(NOT uniquely decodable!!)

瞬時復号可能でない例
(Ex. not instantaneously decodable)

$$S = \{a, b, c\}$$
$$T = \{0, 1\}$$

$$\mathcal{C} : \begin{cases} a \longmapsto 0 \\ b \longmapsto 01 \\ c \longmapsto 11 \end{cases}$$

- **Uniquely decodable (一意復号可能ではある)**
- **“011$\cdots$” $\Longrightarrow$ “ac$\cdots$” or “bc$\cdots$” ?**
  (“0111”$\Longrightarrow$“bc”, “01111”$\Longrightarrow$“acc”)
  $\longrightarrow$ NOT instantaneously decodable
  (瞬時復号可能でない)

瞬時復号可能な例
(Ex. instantaneously decodable)

$$S = \{a, b, c\}$$
$$T = \{0, 1\}$$

$$\mathcal{C} : \begin{cases} a \longmapsto 0 \\ b \longmapsto 10 \\ c \longmapsto 11 \end{cases}$$

$\longrightarrow$ **How can one distinguish**
instantaneously decodable codes
by looking only at $\mathcal{C}(S) = \{0, 10, 11\} \subset T^+$ **?**

瞬時符号の性質
(Properties of instantaneously decodable codes)

- $\mathcal{C}$: instant. decodable $\Longrightarrow$ $\mathcal{C}$: uniq. decodable

- $\mathcal{C}$: instant. decodable
$$\Longleftrightarrow \mathcal{C} \text{ : prefix code (語頭符号)}$$
$$(\mathcal{C}(s') = \mathcal{C}(s)\boldsymbol{x} \Longrightarrow s' = s, \boldsymbol{x} = \varepsilon)$$

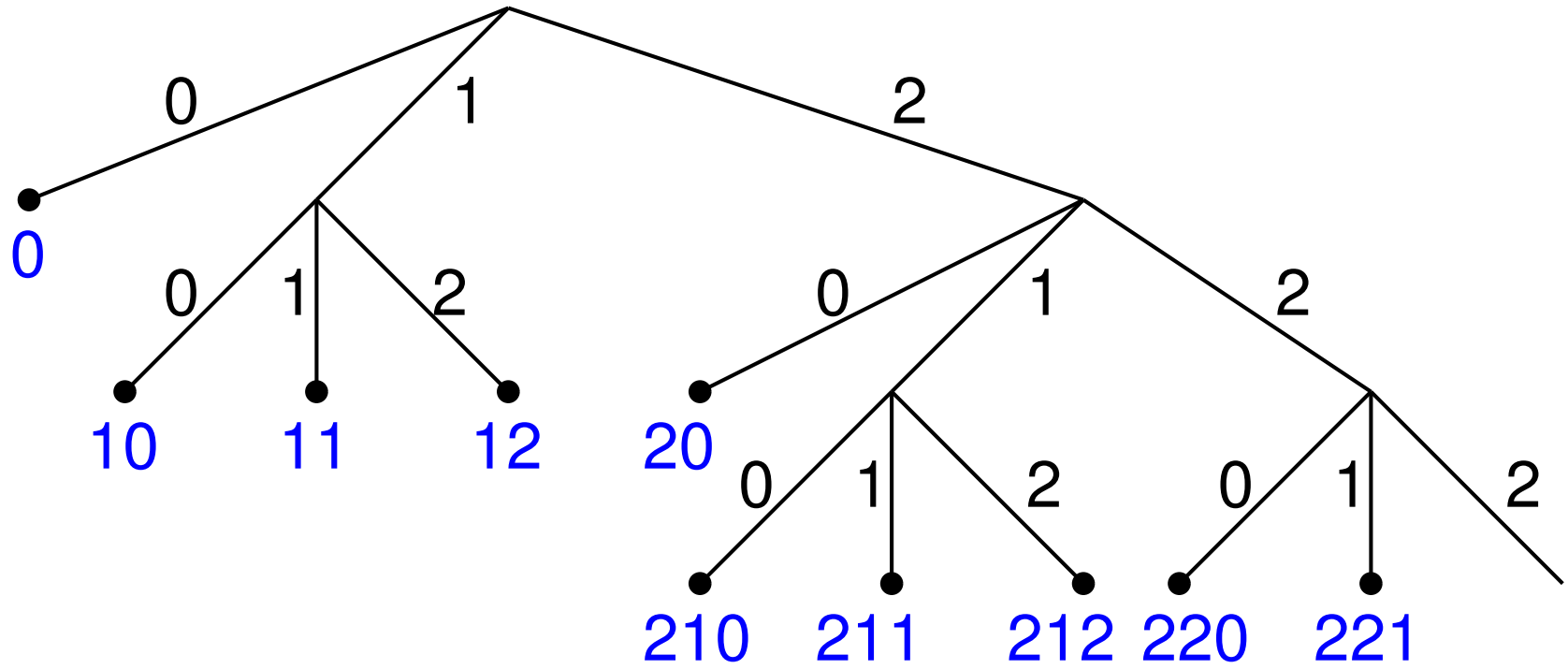
瞬時符号の作り方
(How to construct instant. decodable codes)

符号木 (code tree) を考えよう

符号木 (code tree)

Ex. $\mathsf{T} = \{0, 1, 2\}$

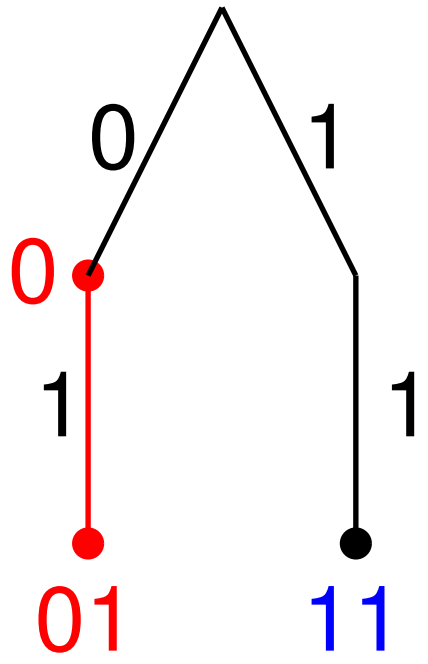
$\mathcal{C}(S) = \{0, 10, 11, 12, 20, 210, 211, 212, 220, 221\}$



符号木と瞬時復号可能性
(Code trees and instantaneous decodablity)

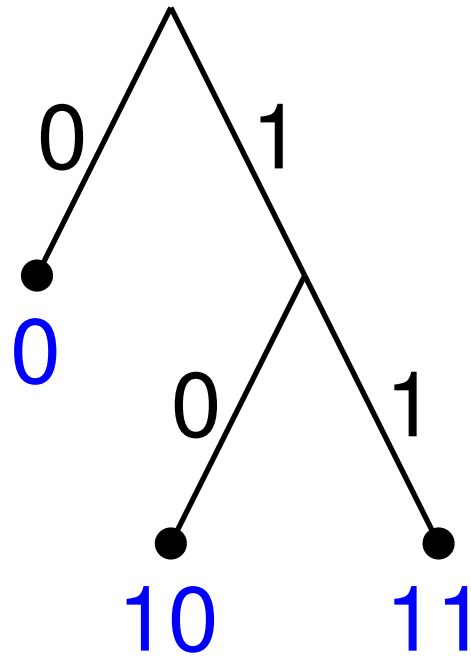
Ex. $\mathsf{T} = \{0, 1\}$

$\mathcal{C}(S) = \{0, 01, 11\}$



not instant. decodable

$\mathcal{C}(S) = \{0, 10, 11\}$



instant. decodable

瞬時符号の効率

(Efficency of instantaneously decodable codes)

瞬時符号という条件を満たしつつ、
出来るだけ効率良くしたい

(Under the condition to be instant. decodable,
make it as efficient as possible.)

The list of the lengths of the code-words

$$(|\mathcal{C}(s)|)_{s\in S} = (|\mathcal{C}(s_1)|, |\mathcal{C}(s_2)|, \ldots, |\mathcal{C}(s_k)|)$$

is to be as "small" as possible.

Kraft の不等式 (Kraft's inequality)

$S = \{s_1, \ldots, s_k\}$, $\quad \#T = r$ **(r-ary code)**

For a sequence $(\ell_1, \ldots, \ell_k)$ **of natural numbers,**

$\exists$ **an** r**-ary instant. decodable code** $\mathcal{C}$
with $|\mathcal{C}(s_i)| = \ell_i$ $(\forall i)$

$$\Longleftrightarrow \sum_{i=1}^{k} \frac{1}{r^{\ell_i}} \leq 1$$

instant. decodable $\Longrightarrow$ uniq. decodable

(the converse is not true)

(Uniq. decodability is a weaker condition.)

If we allow uniq. decodable codes,

can we make the list of lengths more small ?

$\longrightarrow$ No!!

(For any uniq. decodable code,

there exists a instant. decodable code

with the same list of lengths.)

McMillan の不等式 (McMillan's inequality)

$S = \{s_1, \ldots, s_k\}$, $\quad \#T = r$ **(r-ary code)**

For a sequence $(\ell_1, \ldots, \ell_k)$ **of natural numbers,**

$\exists$ **an** r**-ary uniq. decodable code** $\mathcal{C}$ **with** $|\mathcal{C}(s_i)| = \ell_i$ $(\forall i)$

$$\Longleftrightarrow \sum_{i=1}^{k} \frac{1}{r^{\ell_i}} \leq 1$$

母関数 (generating functions)

Construct a function from a sequence (a_n)

$\longrightarrow$ use of analytic method

- $\sum_{n\geq 0} a_n X^n$: **power series (usual type)**
- $\sum_{n\geq 0} \frac{a_n}{n!} X^n$: **exponential type**
- $\sum_{n\geq 1} \frac{a_n}{n} X^n$: **logarithmic type**
- $\sum_{n\geq 1} \frac{a_n}{n^s}$: **Dirichlet series**

Ex. $k = 2$

source length 0:

ε	X^0
	1

source length 1:

w_1	X^{ℓ_1}
w_2	X^{ℓ_2}
	$X^{\ell_1} + X^{\ell_2}$

source length 2:

w_1w_1	$X^{\ell_1}X^{\ell_1} = X^{2\ell_1}$
w_1w_2	$X^{\ell_1}X^{\ell_2} = X^{\ell_1+\ell_2}$
w_2w_1	$X^{\ell_2}X^{\ell_1} = X^{\ell_1+\ell_2}$
w_2w_2	$X^{\ell_2}X^{\ell_2} = X^{2\ell_2}$
	$(X^{\ell_1} + X^{\ell_2})^2$

モールス符号では 1 文字のための符号長が区々
(Various length for each character in Morse code)

⟵ 頻度の高い文字は短く、低い文字は長く
(Shorter if frequent, longer if rare)

⟶ 頻度まで考慮して符号長の期待値を短く
(Shorten the expectation length)

· · · 頻度 (出現確率) も考慮した符号効率の定式化
(Formulate efficiency considering frequency)

「情報源 alphabet と頻度との組」が符号化の対象
(The "source" should mean
a pair of the source alphabet and frequency.)

モールス符号では 1 文字のための符号長が区々
(Various length for each character in Morse code)

⟵ 頻度の高い文字は短く、低い文字は長く
(Shorter if frequent, longer if rare)

⟶ 頻度まで考慮して符号長の期待値を短く
(Shorten the expectation length)

· · · 頻度 (出現確率) も考慮した符号効率の定式化
(Formulate efficiency considering frequency)

「情報源 alphabet と頻度との組」が符号化の対象
(The "source" should mean
a pair of the source alphabet and frequency.)

モールス符号では 1 文字のための符号長が区々
(Various length for each character in Morse code)

$\longleftarrow$ 頻度の高い文字は短く、低い文字は長く
(Shorter if frequent, longer if rare)

$\longrightarrow$ 頻度まで考慮して符号長の期待値を短く
(Shorten the expectation length)

$\cdots$ 頻度 (出現確率) も考慮した符号効率の定式化
(Formulate efficiency considering frequency)

「情報源 alphabet と頻度との組」が符号化の対象
(The "source" should mean
a pair of the source alphabet and frequency.)

情報源符号化の定式化・続
(Formulation of source coding: continued)

S : source alphabet (a finite set)

$\mathrm{P} : S \longrightarrow [0,1] \subset \mathbf{R}$: **生起確率**
(**occurence probability**, $\sum_{s \in S} \mathrm{P}(s) = 1$)

情報源 (**source**) $\mathcal{S} := (S, \mathrm{P})$
文字 $s \in S$ **を確率** $\mathrm{P}(s)$ **で次々と発生**
(generates symbols $s \in S$
with probability $\mathrm{P}(s)$ successively)
$\longrightarrow w \in S^+$ **を発生**
(generates a sequence $w \in S^+$)

Here we assume the following property for simplicity:

(ここでの) 仮定：情報源の定常・無記憶性
(Assumption : stationary, memoryless source)

> **各 $s \in S$ の生起確率 $\mathrm{P}(s)$ は、s のみで決まり、先立って発生した文字に依らない。**
> **The occurence probability $\mathrm{P}(s)$ of $s \in S$ depends only on s, not on preceding symbols.**

情報源符号化の定式化・続
(Formulation of source coding: continued)

T : code alphabet (a finite set)

(typically $\mathsf{T} = \{0, 1\}$)

$\mathcal{C} : S \longrightarrow \mathsf{T}^+$: a code

$\longrightarrow$ 文字列を並べて $\mathcal{C}^* : S^* \longrightarrow \mathsf{T}^*$ に延長

(extended by concatnation)

$\mathrm{L}(\mathcal{C}) := \sum_{s \in S} \mathrm{P}(s)|\mathcal{C}(s)|$: $\mathcal{C}$ の平均符号長

(average code-word-length)

符号への要請 (Requirement for good codes)

- **一意符号 (uniquely decodable code):**

$$\mathcal{C}^* : S^* \longrightarrow \mathrm{T}^* : \textbf{injective (単射)}$$

- **瞬時符号 (instantaneously decodable code):**

$$\mathcal{C}^*(\boldsymbol{x}) = \mathcal{C}(s)\boldsymbol{w} \Longrightarrow \boldsymbol{x} = s\boldsymbol{y}$$

  $\cdots$ **These do not depend on** P**.**

- **効率が良い (efficient) :** $\mathrm{L}(\mathcal{C})$ **is small.**
  $\cdots$ **This depends on** P**.**

Taking the occurence probability P into consideration,

construct a code with small average length.

$\longrightarrow$ **Huffman code**

Taking the occurence probability P
into consideration,

construct a code with small average length.

$\longrightarrow$ **Huffman code**

平均符号長の小さい符号の構成 (Huffman code)

(Construction of a code $\mathcal{C}$ with small $\mathrm{L}(\mathcal{C})$)

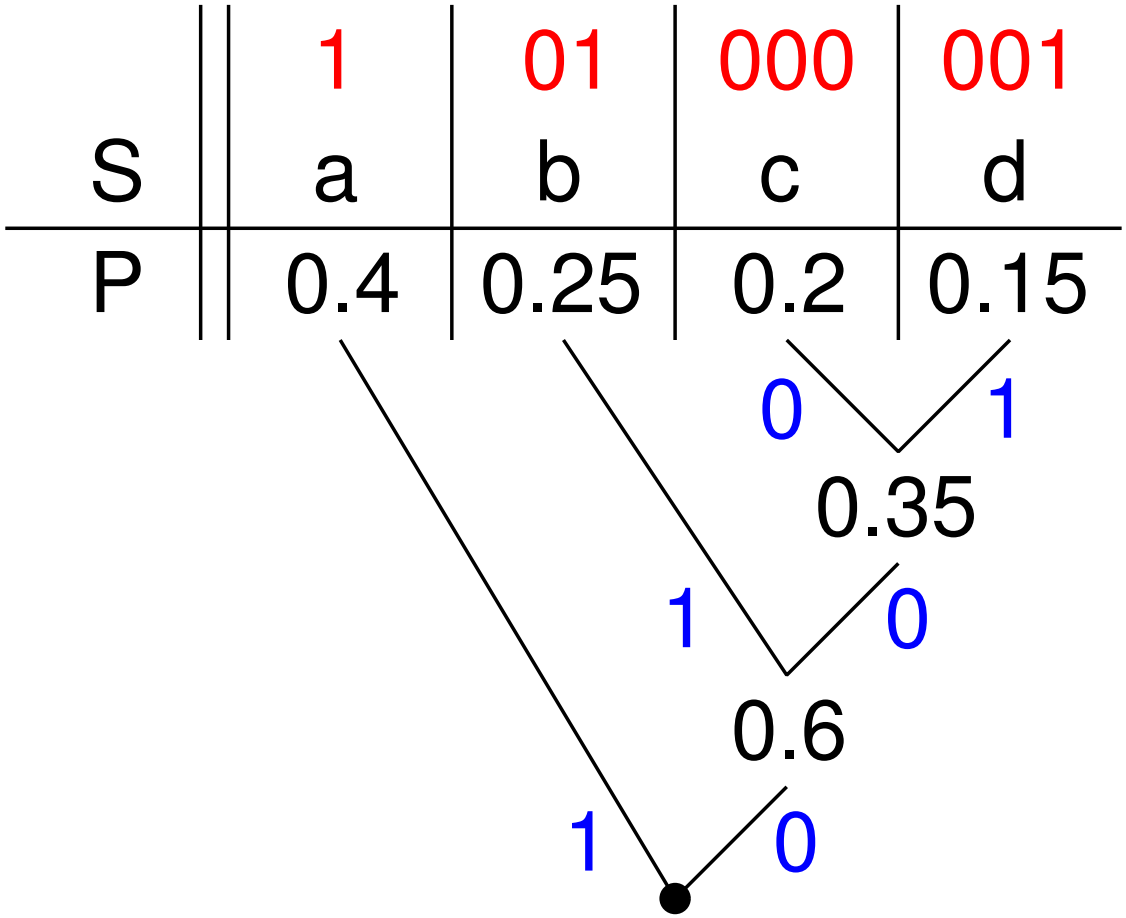
Ex. $\#S = 4 = 2^2$, average length: $1.95 (< 2)$

S	a	b	c	d
P	0.4	0.25	0.2	0.15

平均符号長の小さい符号の構成 (Huffman code)

(Construction of a code $\mathcal{C}$ with small $\mathrm{L}(\mathcal{C})$)

Ex. $\#S = 4 = 2^2$, **average length:** $1.95 (< 2)$



Huffman code

- **平均符号長 $\mathrm{L}(\mathcal{C})$ の最小値を実現**
  (Attain the minimum avarage length $\mathrm{L}(\mathcal{C})$)
  $\cdots$ 最適符号 (optimal code)
- **各文字の生起確率 $\mathrm{P}(s)$ の**
  “ばらつき” が大きいほど効果的
  (More effective
  if $\mathrm{P}(s)$’s are more “scattering”)
- **弱点: 予め生起確率が判らないと**
  符号を構成できない
  (Weak point: must know $\mathrm{P}(s)$’s in advance)

$\#S = 2$ **だったら、**
どうやっても (生起確率に関わらず) $\mathrm{L}(\mathcal{C}) = 1$ **か ?**
(If $\#S = 2$**, must one have** $\mathrm{L}(\mathcal{C}) = 1$ **?)**

何かうまい手はないか ?
(Is there a good way to improve ?)

$\longrightarrow$ **Consider extended sources (拡大情報源)**

$\#S = 2$ **だったら、**
どうやっても (生起確率に関わらず) $\mathrm{L}(\mathcal{C}) = 1$ **か ?**
(If $\#S = 2$**, must one have** $\mathrm{L}(\mathcal{C}) = 1$ **?)**

何かうまい手はないか ?
(Is there a good way to improve ?)

$\longrightarrow$ **Consider extended sources (拡大情報源)**

拡大情報源 (Extended source)

情報源 $\mathcal{S} = (S, P)$ に対し、
“n 文字づつまとめた情報源” $\mathcal{S}^n$ を考える
(For the source $\mathcal{S} = (S, P)$, consider
the source $\mathcal{S}^n$ “packed every n symbols”)

$$\mathcal{S}^n := (S^n, P^{\otimes n})$$

$$S^n = S \times \cdots \times S = \{s_{i_1} \dots s_{i_n} \,|\, s_{i_j} \in S\}$$

$$P^{\otimes n} : S^n \longrightarrow [0,1] \subset \mathbf{R}$$

$$s_{i_1} \dots s_{i_n} \longmapsto P(s_{i_1}) \cdots P(s_{i_n})$$

$\longrightarrow$ **Encode this source $\mathcal{S}^n$.**

問題: 次の生起確率を持つ情報源

$\mathcal{S} = (S, P), S = \{a, b\}$ **について、**

S	a	b
P	0.8	0.2

(1) 2 **次の拡大情報源** $\mathcal{S}^2 = (S^2, P^{\otimes 2})$
に対する Huffman 符号 C_2 **を構成し、**
“1 文字当たりの平均符号長”
$L(C_2)/2$ **を求めよ。**

(2) 3 **次の拡大情報源** $\mathcal{S}^3 = (S^3, P^{\otimes 3})$
に対しても同様の計算をせよ。