情報源符号化の定式化・続
(Formulation of source coding: continued)

S : source alphabet (a finite set)

$P : S \longrightarrow [0,1] \subset \mathbf{R}$: **生起確率**
(**occurence probability**, $\sum_{s \in S} P(s) = 1$)

情報源 (**source**) $\mathcal{S} := (S, P)$
文字 $s \in S$ を確率 $P(s)$ で次々と発生
(generates symbols $s \in S$
with probability $P(s)$ successively)
$\longrightarrow w \in S^+$ を発生
(generates a sequence $w \in S^+$)

Here we assume the following property for simplicity:

(ここでの) 仮定：情報源の定常・無記憶性
(Assumption : stationary, memoryless source)

> **各 $s \in S$ の生起確率 $\mathrm{P}(s)$ は、s のみで決まり、先立って発生した文字に依らない。**
> **The occurence probability $\mathrm{P}(s)$ of $s \in S$ depends only on s, not on preceding symbols.**

情報源符号化の定式化・続
(Formulation of source coding: continued)

T : code alphabet (a finite set)

(typically $T = \{0, 1\}$)

$\mathcal{C} : S \longrightarrow T^+$: a code

$\longrightarrow$ 文字列を並べて $\mathcal{C}^* : S^* \longrightarrow T^*$ に延長

(extended by concatnation)

$$L(\mathcal{C}) := \sum_{s \in S} P(s)|\mathcal{C}(s)|$$: $\mathcal{C}$ の**平均符号長**

(**average code-word-length**)

符号への要請 (Requirement for good codes)

- **一意符号 (uniquely decodable code):**

$$\mathcal{C}^* : \mathcal{S}^* \longrightarrow \mathrm{T}^* : \textbf{injective (単射)}$$

- **瞬時符号 (instantaneous decodable code):**

$$\mathcal{C}^*(\boldsymbol{x}) = \mathcal{C}(s)\boldsymbol{w} \Longrightarrow \boldsymbol{x} = s\boldsymbol{y}$$

$\cdots$ **These do not depend on** P**.**

- **効率が良い (efficient) :** $\mathrm{L}(\mathcal{C})$ **is small.**

$\cdots$ **This depends on** P**.**

Taking the occurence probability P into consideration,

construct a code with small average length.

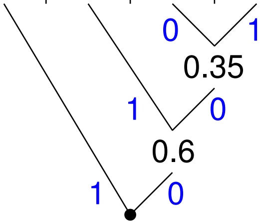
$\longrightarrow$ **Huffman code**

平均符号長の小さい符号の構成 (Huffman code)

(Construction of a code $\mathcal{C}$ with small $\mathrm{L}(\mathcal{C})$)

Ex. $\#S = 4 = 2^2$, **average length:** $1.95 (< 2)$

	1	01	000	001
S	a	b	c	d
P	0.4	0.25	0.2	0.15



Huffman code

- **平均符号長** $L(\mathcal{C})$ **の最小値を実現**
  (Attain the minimum avarage length $L(\mathcal{C})$**)**
  $\cdots$ **最適符号 (optimal code)**
- **各文字の生起確率** $P(s)$ **の**
  "ばらつき" が大きいほど効果的
  (More effective
  if $P(s)$**'s are more "scattering")**
- **弱点: 予め生起確率が判らないと**
  符号を構成できない
  (Weak point: must know $P(s)$**'s in advance)**

$\#S = 2$ **だったら、**
どうやっても (生起確率に関わらず) $\mathrm{L}(\mathcal{C}) = 1$ **か ?**
(If $\#S = 2$**, must one have** $\mathrm{L}(\mathcal{C}) = 1$ **?)**

何かうまい手はないか ?
(Is there a good way to improve ?)

$\longrightarrow$ **Consider extended sources (拡大情報源)**

拡大情報源 (Extended source)

情報源 $\mathcal{S} = (S, P)$ に対し、
"n 文字づつまとめた情報源" $\mathcal{S}^n$ を考える
(For the source $\mathcal{S} = (S, P)$, consider
the source $\mathcal{S}^n$ "packed every n symbols")

$\mathcal{S}^n := (S^n, P^{\otimes n})$

$$S^n = S \times \cdots \times S = \{s_{i_1} \ldots s_{i_n} \,|\, s_{i_j} \in S\}$$

$$P^{\otimes n} : S^n \longrightarrow [0, 1] \subset \mathbf{R}$$

$$s_{i_1} \ldots s_{i_n} \longmapsto P(s_{i_1}) \cdots P(s_{i_n})$$

$\longrightarrow$ **Encode this source $\mathcal{S}^n$.**

問題: 次の生起確率を持つ情報源

$\mathcal{S} = (S, P), S = \{a, b\}$ **について、**

S	a	b
P	0.8	0.2

(1) 2 **次の拡大情報源** $\mathcal{S}^2 = (S^2, P^{\otimes 2})$ **に対する Huffman 符号** $\mathcal{C}_2$ **を構成し、**
“1 文字当たりの平均符号長” $L(\mathcal{C}_2)/2$ **を求めよ。**

(2) 3 **次の拡大情報源** $\mathcal{S}^3 = (S^3, P^{\otimes 3})$ **に対しても同様の計算をせよ。**

By considering the code $\mathcal{C}_n$ for the extended source $\mathcal{S}^n = (S^n, P^{\otimes n})$ of degree n,

the average length

$$\frac{L(\mathcal{C}_n)}{n}$$

per each symbol can be decreased generally

(though the code turns more complex).

For the code $\mathcal{C}_n$

for the extended source $\mathcal{S}^n = (S^n, P^{\otimes n})$

of degree n,

what is the infimum of $\dfrac{L(\mathcal{C}_n)}{n}$?

$\longrightarrow$ **It will not be smaller than**

"the information content"

which the source $\mathcal{S}$ has properly.

the notion of "entropy(エントロピー)"

“the information content (情報の量)”

「或る事象 P が起こる」という “情報の価値” は、
どう評価したら良いか？

How should one quantify
“the information content”
that an event P occurs ?

“the information content (情報の量)”

Basic idea:

The value to know

that an event with probability $\frac{1}{4}$ occurs

is the same as the one to know

that two events with probability $\frac{1}{2}$ occur.

(確率 $\frac{1}{4}$ で起きる出来事を知ることの価値は、

確率 $\frac{1}{2}$ で起きる出来事を 2 つ知るのと同じ)

$\longrightarrow$ **“the content” is double (“情報の量” が 2 倍)**

「事象 P が起こる」という "情報の量" $I(P)$

("the information content" $I(P)$ that an event P occurs)

要請 (Requirement):

(1) depends only on the occurence probability p

$$\longrightarrow I(p) := I(P)$$

(2) for two independent events P_1, P_2,

$$I(P_1 \wedge P_2) = I(P_1) + I(P_2)$$

$$\longrightarrow I(p_1 p_2) = I(p_1) + I(p_2)$$

(3) $I : (0, 1] \longrightarrow \mathbf{R}_{\geq 0}$ **: continuous (not const.** 0**)**

$$\longrightarrow \boxed{I(p) = C \log \frac{1}{p} = -C \log p \quad (C > 0)}$$

「事象 P が起こる」という "情報の量" $I(P)$

("the information content" $I(P)$ that an event P occurs)

$$I(p) = C \log \frac{1}{p} = -C \log p \quad (C > 0)$$

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the choice of the constant C

$\longleftrightarrow$ the choice of the base of $\log$

$\longleftrightarrow$ the choice of the unit of "information"

Usually we choose 2 as the base; $I(\frac{1}{2}) := 1$.

$\longrightarrow$ the unit of "information": bit (binary digit)

情報源のエントロピー (the entropy of a source)

the expected value of the information from the source $\mathcal{S} = (S, P)$ per each symbol:

$$\boxed{H(\mathcal{S}) := \sum_{s \in S} P(s) I(P(s))}$$

: the entropy of the source $\mathcal{S}$

$$S = \{s_1, \ldots, s_k\}, \quad P(s_i) = p_i$$

$$\longrightarrow H(\mathcal{S}) = \sum_{i=1}^{k} p_i \log \frac{1}{p_i} = -\sum_{i=1}^{k} p_i \log p_i$$

情報源のエントロピー (the entropy of a source)

In particular when $\#S = 2$ **(**$S = \{$**yes**, **no**$\}$**),**

- **the probability to occur:** p
- **the probability not to occur:** $1 - p =: \overline{p}$

$$H(p) := H(\mathcal{S}) = p \log \frac{1}{p} + \overline{p} \log \frac{1}{\overline{p}}$$

: binary entropy function

For the code $\mathcal{C}_n$

for the extended source $\mathcal{S}^n = (S^n, P^{\otimes n})$

of degree n,

what is the infimum of $\dfrac{L(\mathcal{C}_n)}{n}$?

$\longrightarrow$ **It will not be smaller than**

the entropy $H(\mathcal{S})$ of $\mathcal{S}$.

$\longrightarrow$ **For a code $\mathcal{C}$,**

first compare the avarage length $L(\mathcal{C})$

with the entropy $H(\mathcal{S})$.

Theorem

$\mathcal{S} = (S, P)$ **: a source**

$\mathcal{C}$ **: a uniq. (or instant.) decodable code for** $\mathcal{S}$

$$\Longrightarrow \qquad \boxed{L(\mathcal{C}) \geq H(\mathcal{S})}$$

(Here, the base of $\log$ **is chosen as** $r := \#T$**.)**

$\eta := \dfrac{H(\mathcal{S})}{L(\mathcal{C})}$ **: the efficiency (効率) of** $\mathcal{C}$

$\overline{\eta} = 1 - \eta$ **: the redundancy (冗長度) of** $\mathcal{C}$

Kraft の不等式 (Kraft's inequality)

$S = \{s_1, \ldots, s_k\}, \qquad \#T = r$ **(r-ary code)**

For a sequence $(\ell_1, \ldots, \ell_k)$ **of natural numbers,**

$\exists$ **an r-ary <u>instant. decodable code</u>** $\mathcal{C}$
with $|\mathcal{C}(s_i)| = \ell_i$ **($\forall i$)**

$$\Longleftrightarrow \sum_{i=1}^{k} \frac{1}{r^{\ell_i}} \leq 1$$

Lemma

$x_i, y_i > 0 \ (i = 1, \ldots, k)$

$$\sum_{i=1}^{k} x_i = \sum_{i=1}^{k} y_i = 1$$

$$\Longrightarrow \qquad \sum_{i=1}^{k} x_i \log \frac{1}{x_i} \leq \sum_{i=1}^{k} x_i \log \frac{1}{y_i}$$

(Equality iff $\forall i : x_i = y_i$)