“the information content (情報の量)”

「或る事象 P が起こる」という “情報の価値” は、
どう評価したら良いか ？

How should one quantify
“the information content”
that an event P occurs ?

「事象 P が起こる」という “情報の量” $I(P)$

(“the information content” $I(P)$ that an event P occurs)

要請 (Requirement):

(1) depends only on the occurence probability p

$$\longrightarrow I(p) := I(P)$$

(2) for two independent events P_1, P_2,

$$I(P_1 \wedge P_2) = I(P_1) + I(P_2)$$

$$\longrightarrow I(p_1 p_2) = I(p_1) + I(p_2)$$

(3) $I : (0,1] \longrightarrow \mathbf{R}_{\geq 0}$: continuous (not const. 0)

$$\longrightarrow \boxed{I(p) = C \log \frac{1}{p} = -C \log p \quad (C > 0)}$$

「事象 P が起こる」という “情報の量” $\mathrm{I}(\mathrm{P})$

(“the information content” $\mathrm{I}(\mathrm{P})$ that an event P occurs)

$$\mathrm{I}(\mathrm{p}) = \mathrm{C}\log\frac{1}{\mathrm{p}} = -\mathrm{C}\log\mathrm{p} \quad (\mathrm{C} > 0)$$

---

the choice of the constant C

$\longleftrightarrow$ the choice of the base of $\log$

$\longleftrightarrow$ the choice of the unit of “information”

Usually we choose 2 as the base; $\mathrm{I}(\frac{1}{2}) := 1$.

$\longrightarrow$ the unit of “information”: bit (binary digit)

情報源のエントロピー (the entropy of a source)

the expected value of the information from the source $\mathcal{S} = (S, P)$ per each symbol:

$$\boxed{H(\mathcal{S}) := \sum_{s \in S} P(s) I(P(s))}$$

: the entropy of the source $\mathcal{S}$

$$S = \{s_1, \ldots, s_k\}, \quad P(s_i) = p_i$$

$$\longrightarrow H(\mathcal{S}) = \sum_{i=1}^{k} p_i \log \frac{1}{p_i} = -\sum_{i=1}^{k} p_i \log p_i$$

For the code $\mathcal{C}_n$

for the extended source $\mathcal{S}^n = (\mathrm{S}^n, \mathrm{P}^{\otimes n})$

of degree n,

what is the infimum of $\dfrac{\mathrm{L}(\mathcal{C}_n)}{n}$?

$\longrightarrow$ **It will not be smaller than**

the entropy $\mathrm{H}(\mathcal{S})$ of $\mathcal{S}$.

$\longrightarrow$ **For a code $\mathcal{C}$,**

first compare the avarage length $\mathrm{L}(\mathcal{C})$

with the entropy $\mathrm{H}(\mathcal{S})$.

Theorem

$\mathcal{S} = (\mathrm{S}, \mathrm{P})$: **a source**

$\mathcal{C}$: **a uniq. (or instant.) decodable code for** $\mathcal{S}$

$$\Longrightarrow \qquad \boxed{\mathrm{L}(\mathcal{C}) \geq \mathrm{H}(\mathcal{S})}$$

(Here, the base of $\log$ **is chosen as** $\mathrm{r} := \#\mathsf{T}$**.)**

$\eta := \dfrac{\mathrm{H}(\mathcal{S})}{\mathrm{L}(\mathcal{C})}$: **the efficiency (効率) of** $\mathcal{C}$

$\overline{\eta} = 1 - \eta$: **the redundancy (冗長度) of** $\mathcal{C}$

Kraft の不等式 (Kraft's inequality)

$S = \{s_1, \ldots, s_k\}$, $\quad \#T = r$ **(r-ary code)**

For a sequence $(\ell_1, \ldots, \ell_k)$ **of natural numbers,**

$\exists$ **an** r**-ary instant. decodable code** $\mathcal{C}$
with $|\mathcal{C}(s_i)| = \ell_i$ $(\forall i)$

$$\Longleftrightarrow \sum_{i=1}^{k} \frac{1}{r^{\ell_i}} \leq 1$$

Lemma

$x_i, y_i > 0 \ (i = 1, \ldots, k)$

$$\sum_{i=1}^{k} x_i = \sum_{i=1}^{k} y_i = 1$$

$$\Longrightarrow \qquad \sum_{i=1}^{k} x_i \log \frac{1}{x_i} \leq \sum_{i=1}^{k} x_i \log \frac{1}{y_i}$$

(Equality iff $\forall i : x_i = y_i$)

$$\boxed{\mathrm{L}(\mathcal{C}) \geq \mathrm{H}(\mathcal{S})}$$

- **Can $\mathrm{L}(\mathcal{C}) = \mathrm{H}(\mathcal{S})$ be attained for some $\mathcal{C}$?**
- $\inf_{\mathcal{C}} \mathrm{L}(\mathcal{C}) = \mathrm{H}(\mathcal{S})$ **?**

Though Huffman code is optimal,
the estimate of $\mathrm{L}(\mathcal{C})$ from above is difficult.

$\longrightarrow$ **Shannon-Fano code**
(use of Kraft-McMillan inequality)

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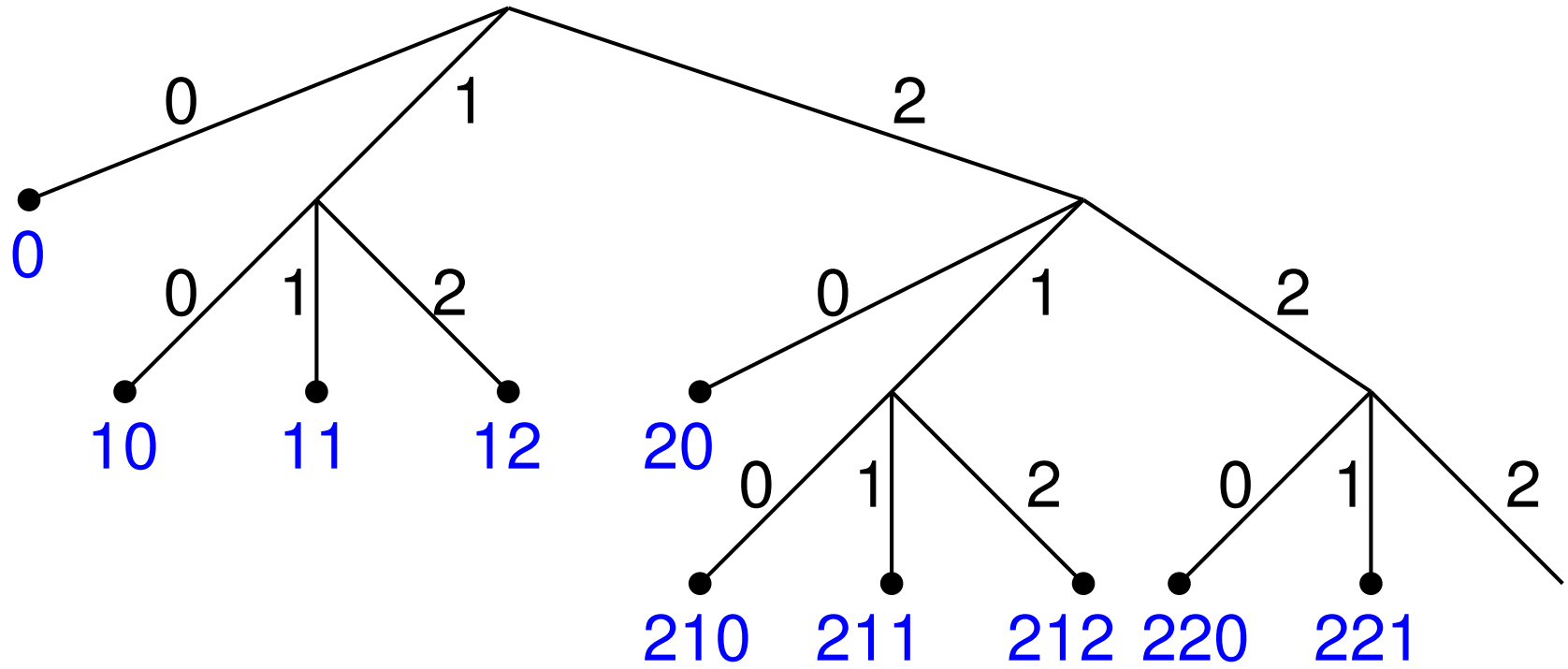
符号木 (code tree)

Ex. $\mathsf{T} = \{0, 1, 2\}$

$\mathcal{C}(S) = \{0, 10, 11, 12, 20, 210, 211, 212, 220, 221\}$



Shannon-Fano codes

For $S = \{s_1, \ldots, s_k\}$, $P(s_i) = p_i$, $\#T = r$,

put $\ell_i := \left\lceil \log_r \left(\frac{1}{p_i} \right) \right\rceil$

$$\longrightarrow \sum_{i=1}^{k} \frac{1}{r^{\ell_i}} \leq 1$$

$\longrightarrow \exists$ **an** r**-ary instant. decodable code** $\mathcal{C}$ **with** $|\mathcal{C}(s_i)| = \ell_i$ **for all** i

For this $\mathcal{C}$**, we have** $H(\mathcal{S}) \leq L(\mathcal{C}) < 1 + H(\mathcal{S})$**.**

Theorem

$\mathcal{S} = (S, \mathrm{P})$: **a source**

$\mathcal{C}$: **an optimal code for** $\mathcal{S}$

$$\Longrightarrow \qquad \boxed{\mathrm{H}(\mathcal{S}) \leq \mathrm{L}(\mathcal{C}) < 1 + \mathrm{H}(\mathcal{S})}$$

(Here, the base of $\log$ **is chosen as** $\mathrm{r} := \#\mathsf{T}$**.)**

Shannon's Noiseless Coding Theorem

$\mathcal{S} = (S, P)$: **a source**

$\mathcal{S}^n = (S^n, P^{\otimes n})$

: **the extended source of** $\mathcal{S}$ **of degree** n

$\mathcal{C}_n$: **an optimal code for** $\mathcal{S}^n$

$$\Longrightarrow \qquad \boxed{\lim_{n\to\infty} \frac{L(\mathcal{C}_n)}{n} = H(\mathcal{S})}$$

(Here, the base of $\log$ **is chosen as** $r := \#T$**.)**

情報源を効率良く符号化する話は一段落

Here we stop saying about the source coding and go to the next topic.

情報通信を行なう際の要請

(What is required in communication ?)

- **効率的に (efficiently)**
  $\longrightarrow$ 情報理論 (Information Theory)
- **確実に (certainly)**
  $\longrightarrow$ 符号理論 (Coding Theory)
- **安全に (safely)**
  $\longrightarrow$ 暗号理論 (Cryptography)

情報通信にはノイズ (雑音) が妨げとなる
Noises are obstructive in communication.

雑音の入る通信路を介して情報通信を行なう際、
通信途中での誤りに如何に対処するか?
When trying to communicate
with noisy channel,
how can one overcome the errors ?

誤りに対処しつつ、如何に効率的に伝達するか?
Dealing with the errors,
how efficiently can one communicate ?

⟶ Coding theory(符号理論)
• Error-correcting codes(誤り訂正符号)

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$\longrightarrow$ **Coding theory(符号理論)**
- **Error-correcting codes(誤り訂正符号)**

誤り訂正符号 (お話)(Error-correcting codes)

How to overcome errors occurring in channels:

- **Physical technology: Error suppression**
  誤りの発生を抑える (物理技術による対処)
  (導線の高品質化・ノイズの遮蔽・等々)
- **Social technology: The thought of "fail-safe"**
  誤りが起きても致命的にならないように
  (フェイルセーフの発想・社会技術)
- **Mathematical technology: Error correction**
  多少の誤りなら検出・訂正できる仕組み
  $\longrightarrow$ 数理技術により実現 (誤り訂正符号)

誤り訂正符号 (お話)(Error-correcting codes)

情報通信中に誤り (らしきこと) に出遭ったら ?
How should we do when we meet
a (suspicious) error in communication ?

例: 「じゃあヨツバ駅で待ち合わせね」
"We will meet at Yotsuba station."

- 聞き直す (より安全なプロトコルの採用)
  Ask again (more secure protocol)
- 見当を付ける (誤りの自動訂正)
  Make a guess (error-correcting)

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例: 「じゃあヨツバ駅で待ち合わせね」

"We will meet at Yotsuba station."

$\longrightarrow$ きっとヨツヤ駅だろう

It should be Yotsuya station.

- Why can one be aware of the error?

  $\longrightarrow$ No station has the name "Yotsuba".

- Why can one guess the correct name?

  $\longrightarrow$ No other station has a similar name.

$\longleftarrow$ Only few strings are correct names.

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Use of redundancy to keep certainty

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「誤りの自動訂正」が出来るように
予め適切に冗長性を持たせて通信する

Give some controlled redundancy beforehand
to do automatic error-correction.

- 誤り訂正性能は高く
  (High error-correction ability)
- とは言えなるべく効率的に
  (As efficient as possible)

$\longrightarrow$ 有限体上の線型代数・代数幾何などの利用
(Use of linear algebra and algebraic geometry
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